On the vanishing electron-mass limit in plasma hydrodynamics in unbounded media

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Abstract

We consider the zero-electron-mass limit for the Navier-Stokes-Poisson system in unbounded spatial domains. Assuming smallness of the viscosity coefficient and ill-prepared initial data, we show that the asymptotic limit is represented by the incompressible Navier-Stokes system, with a Brinkman damping, in the case when viscosity is proportional to the electron-mass, and by the incompressible Euler system provided the viscosity is dominated by the electron mass. The proof is based on the RAGE theorem and dispersive estimates for acoustic waves, and on the concept of suitable weak solutions for the compressible Navier-Stokes system.

1 Introduction

Singular limits arise frequently in the process of *model reduction* in fluid mechanics. In this paper we consider the limit of vanishing ratio electron mass/ion mass in a hydrodynamic model for plasma confined to an unbounded spatial domain $\Omega \subset R^3$.

1.1 Equations

For a given (constant) density N_i of positively charged ions, the time evolution of the electron density $n_e = n_e(t,x)$ and the electron velocity $\mathbf{u} = \mathbf{u}(t,x)$ is governed by the system of equations

$$\partial_t n_e + \mathrm{div}_x (n_e \mathbf{u}) = 0, \tag{1.1}$$

$$m_e \Big(\partial_t (n_e \mathbf{u}) + \mathrm{div}_x (n_e \mathbf{u} \otimes \mathbf{u}) \Big) + \nabla_x p(n_e) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + n_e \nabla_x \Phi - m_e \frac{n_e \mathbf{u}}{\tau}, \tag{1.2}$$

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$$\Delta\Phi = n_e - N_i, \tag{1.3}$$

where m_e is the ratio of the electron/ions mass, p is the electron pressure, Φ is the electric potential, τ is the relaxation time, and $\mathbb{S}$ denotes the viscous stress tensor,

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big),\ \mu > 0, \tag{1.4}$$

see Anile and Pennisi [3], Jüngel and Peng [18], [19]. Moreover, we suppose the electron velocity satisfies the slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, [\mathbb{S}(\nabla_x\mathbf{u})\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0, \tag{1.5}$$

and the boundary is electrically insulated,

$$\nabla_x\Phi\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{1.6}$$

As the underlying spatial domain is unbounded, we also prescribe prescribe the far field behavior:

$$\mathbf{u}(x)\to 0,\ \Phi(x)\to 0 \text{ as } |x|\to\infty. \tag{1.7}$$

Our goal is to study the singular limit and identify the limit problem for $m_e \to 0$ under the condition

- $\mu \approx m_e$, or
- $\mu/m_e \to 0$ as $m_e \to 0$.

1.2 Ill-prepared initial data

For $m_e = \varepsilon^2$, $\mu_\varepsilon = \mu/\varepsilon^2$, problem (1.1 - 1.7) is reminiscent of the low Mach (incompressible) limit of the Navier-Stokes system that have been investigated in a number of recent studies, see see the survey papers by Danchin [8], Gallagher [16], Masmoudi [22], and Schochet [25], and the references cited therein. The zero-electron-mass limit for the *inviscid* fluid was treated recetely by Ali et al. [2], see also Chen, Chen and Zhang [6]. In the latter case, it is shown that the systems becomes neutral, meaning $n_e \to N_i$, while the limit velocity field $\mathbf{v}$ satisfies a damped Euler system

$$\mathrm{div}_x\mathbf{v} = 0, \tag{1.8}$$

$$\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla_x\mathbf{v} + \frac{1}{\tau}\mathbf{v} + \nabla_x\Pi = 0, \tag{1.9}$$

supplemented with the impermeability condition

$$\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{1.10}$$

In [2], [6], the authors consider the periodic boundary conditions and *well-prepared* initial data

$$n_e(0,\cdot) = N_i + \varepsilon^2 N_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \mathrm{div}_x\mathbf{u}_{0,\varepsilon} = 0.$$

In this paper, we focus on the *ill-prepared* data, specifically,

$$n_e(0,\cdot) = N_i + \varepsilon N_{0,\varepsilon},\ \{N_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2\cap L^\infty(\Omega), \tag{1.11}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega;R^3). \tag{1.12}$$

In particular, the gradient part of the velocity field will develop fast oscillations in the asymptotic limit $\varepsilon\to 0$.

1.3 Spatial domain

In contrast with [2], we consider the physically relevant (unbounded) domains with boundaries. Similarly to Farwig, Kozono, and Sohr [12], we focus on the class of *uniform* C^3 domains of type (α, β, K). Specifically, for each point of $x_0 \in \partial\Omega$, there is a function $h \in C^3(R^2)$, $\|h\|_{C^3(R^2)} \leq K$, and

$$U_{\alpha,\beta,h} = \{(y, x_3) \ |h(y) - \beta < x_3 < h(y) + \beta, \ |y| < \alpha\}$$

such that, after suitable translation and rotation of the coordinate axes, $x_0 = [0, 0, h(0)]$ and

$$\Omega \cap U_{\alpha,\beta,h} = \{(y, x_3) \ |h(y) - \beta < x_3 < h(y), \ |y| < \alpha\},$$

$$\partial\Omega \cap U_{\alpha,\beta,h} = \{(y, x_3) \ |x_3 = h(y), \ |y| < \alpha\}.$$

Additional hypotheses imposed on the class of domains are stronger for the inviscid limit so we consider the two cases separately.

1.3.1 Hypotheses in the case of constant viscosity

Since our method leans essentially on dispersion of acoustic waves, we suppose that

- the *point spectrum* of the Neumann Laplacian Δ_N in $L^2(\Omega)$ is empty,

in particular, the domain Ω must be unbounded. Although the absence of eigenvalues for the Neumann Laplacian represents, in general, a delicate and highly unstable problem (see Davies and Parnovski [10]), there are numerous examples of such domains - the whole space R^3, the half-space, exterior domains, unbounded strips, tube-like domains and waveguides, see D'Ancona and Racke [9].

1.3.2 Hypotheses in the case of inviscid limit

The absence of eigenvalues for the Neumann Laplacian is apparently not sufficient to carry over the inviscid limit. We need stronger dispersion provided by the so-called $L^1 - L^\infty$ estimates well-known for the acoustic equation in R^3, cf. Section 7.1 below. More specifically, we focus on the class of physically relevant domains represented by *infinite waveguides* in the spirit of D'Ancona and Racke [9]. We suppose that

$$\Omega \subset R^3, \ \Omega = B \times R^L, \ L = 1, 2, 3, \tag{1.13}$$

where

$$B \subset R^{3-L} \text{ is a smooth bounded domain for } L = 1, 2, \ \Omega = R^3 \text{ for } L = 3. \tag{1.14}$$

Obviously, the domains satisfying (1.13), (1.14) belong to the class of uniform C^3 domains of type (α, β, K), and the point spectrum of the Neumann Laplacian is empty. A peculiar feature of the present problem is that propagation of *acoustic waves* is governed by a wave equation of *Klein-Gordon* type (see Section 5), where dispersion is enhanced by the presence of "damping". In particular, we recover the $L^1 - L^\infty$ estimates even in the case of infinite tubes ($L = 1$) under the Neumann boundary conditions, see Section 7.1.

1.4 Asymptotic limit

By analogy with the low Mach number limits, we expect the limit velocity to satisfy the incompressible Navier-Stokes system with a Brinkman type damping if $\mu_\varepsilon = \text{const} > 0$, and the Euler system (1.8 - 1.10) in the inviscid limit $\mu_\varepsilon \to 0$.

In comparison with the low Mach number limit, the main difficulty here is the presence of the extra term

$$\frac{1}{\varepsilon^2} n_e \nabla_x \Phi = \frac{n_e - N_i}{\varepsilon} \nabla_x \Delta_N^{-1} \left[\frac{n_e - N_i}{\varepsilon} \right] + \frac{N_i}{\varepsilon^2} \nabla_x \Phi$$

in the momentum equation (1.2). While the gradient component $\frac{N_i}{\varepsilon^2} \nabla_x \Phi$ can be easily incorporated into the pressure in the limit system, the quantity

$$\frac{n_e - N_i}{\varepsilon} \nabla_x \Delta_N^{-1} \left[\frac{n_e - N_i}{\varepsilon} \right]$$

should "disappear" in the course of the limit process $\varepsilon \to 0$. To achieve this, the dispersive estimates based on the celebrated RAGE theorem will be used.

Another difficulty lies in the fact that the quantity

$$\frac{n_e - N_i}{\varepsilon} \nabla_x \Delta_N^{-1} \left[\frac{n_e - N_i}{\varepsilon} \right]$$

is (known to be) only locally integrable; for global analysis, it must be written in the form

$$\frac{n_e - N_i}{\varepsilon} \nabla_x \Delta_N^{-1} \left[\frac{n_e - N_i}{\varepsilon} \right] = \frac{1}{\varepsilon^2} \left(\mathrm{div}_x \left(\nabla_x \Phi \otimes \nabla_x \Phi \right) - \frac{1}{2} \nabla_x |\nabla_x \Phi|^2 \right),$$

meaning as an element of the dual space $W^{-1,1}$.

Last but not least, we point out that the analysis of the *inviscid* limit leans heavily on the fact that the propagation of acoustic waves is governed by the Klein-Gordon wave equation, yielding effective dispersion on the waveguide like domains specified in Section 1.3.2.

The paper is organized as follows. In Section 2, we introduce the concept of *suitable weak solution* to system (1.1 - 1.7) that proved to be very convenient for studying the inviscid limits, cf. [14]. Section 3 contains the main results. In Section 4, we summarize the uniform bounds independent of the scaling parameter ε. Section 5 is devoted to the acoustic equation and the resulting dispersive estimates. Finally, in Section 6, we show convergence toward the incompressible Navier-Stokes system in the case of non-degenerate viscosity, while Section 7 completes the proof of the inviscid limit.

2 Suitable weak solutions

Motivated by the genera theory developed by Ruggeri and Trovato [24], we assume that the electron pressure p satisfies

$$p \in C[0, \infty) \cap C^2(0, \infty),\ p(0) = 0,\ p'(n) > 0 \text{ for all } n > 0,\ \lim_{n \to \infty} \frac{p'(n)}{n^{2/3}} = p_\infty > 0. \tag{2.1}$$

Next, we introduce the standard *Helmholtz decomposition* of a vector field $\mathbf{v}$,

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}],$$

with

$$\mathbf{H}^{\perp} = \nabla_x \Psi,\ \Delta \Psi = \mathrm{div}_x \mathbf{v} \text{ in } \Omega,\ (\nabla_x \Psi - \mathbf{v}) \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

As shown by Farwig, Kozono and Sohr [12], the linear operator

$$\mathbf{H} \text{ is bounded in } L^2 \cap L^q(\Omega; R^3) \text{ for } 2 < q < \infty, \text{ and in } L^2 + L^q(\Omega) \text{ for } 1 < q \leq 2$$

as soon as Ω is a C^2-domain of type (α, β, K) introduced in Section 1.3. Moreover, the norm of $\mathbf{H}$ in the aforementioned spaces depends solely on the parameters (α, β, K). As a matter of fact, the domains considered in the present paper belong to the higher regularity class C^3 for several technical reasons that will become clear in the course of the proof of the main results.

Following [15] we say that a trio n_e, $\mathbf{u}$, Φ is a *suitable weak solution* to system (1.1 - 1.7) in $(0,T) \times \Omega$, supplement with the initial conditions $n_e(0,\cdot) = n_0$, $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ if:

- the functions n_e, $\mathbf{u}$, Φ belong to the regularity class

$$n_e \geq 0,\ n_e - N_i \in L^\infty(0,T; L^{5/3} + L^2(\Omega)),$$
$$p(n_e) \in L^1(0,T; L^1_{\mathrm{loc}}(\Omega)),$$
$$\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3)),\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$
$$\nabla_x \Phi \in L^\infty(0,T; L^2(\Omega; R^3));$$

- equation of continuity (1.1) is satisfied in the sense of renormalized solutions (see DiPerna and P.-L.Lions [11]),

$$\int_0^T \int_\Omega \Big((b(n_e) + n_e)\, \partial_t \varphi + (b(n_e) + n_e)\, \mathbf{u} \cdot \nabla_x \varphi + \Big(b(n_e) - b'(n_e) n_e \Big) \mathrm{div}_x \mathbf{u} \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.2}$$
$$= - \int_\Omega (b(n_0) + n_0))\, \varphi(0,\cdot) \ \mathrm{d}x$$

  for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$ and any $b \in C^\infty[0,\infty)$, $b' \in C^\infty_c[0,\infty)$;

- momentum equation (1.2), together with the slip boundary condition (1.5), is satisfied in a weak sense,

$$\int_0^T \int_\Omega \Big(m_e n_e \mathbf{u} \cdot \partial_t \varphi + m_e n_e (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p(n_e) \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.3}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi - n_e \nabla_x \Phi \cdot \varphi + \frac{m_e n_e}{\tau} \mathbf{u} \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega n_0 \mathbf{u}_0 \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

  for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{[0,T) \times \partial\Omega} = 0$;

- the electric potential Φ is given by formula

$$\nabla_x \Phi(s,\cdot) = \nabla_x \Phi_0 - \int_0^s \mathbf{H}^{\perp}[n_e \mathbf{u}] \ \mathrm{d}t, \tag{2.4}$$

  where

$$\Delta \Phi_0 = n_0 - N_i \text{ in } \Omega,\ \nabla_x \Phi_0 \cdot \mathbf{n}|_{\partial\Omega} = 0; \tag{2.5}$$

- the *relative entropy inequality*

$$\int_\Omega \left(\frac{1}{2} m_e n_e |\mathbf{u} - \mathbf{U}|^2 + E(n_e, r) + \frac{1}{2} |\nabla_x \Phi|^2 \right) (s, \cdot) \ \mathrm{d}x \tag{2.6}$$

$$+ \int_0^s \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \int_0^s \int_\Omega \frac{m_e}{\tau} n_e |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} m_e n_0 |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + E(n_0, r(0, \cdot)) + \frac{1}{2} |\nabla_x \Phi_0|^2 \right) \ \mathrm{d}x + \int_0^s \mathcal{R}\left(n_e, \mathbf{u}, r, \mathbf{U}\right) \ \mathrm{d}t$$

holds for a.a. $s \in [0, T]$ and all test functions r, $\mathbf{U}$ such that

$$r - N_i \in C_c^\infty([0, T] \times \overline{\Omega}), \ r > 0, \ \mathbf{U} \in C_c^\infty([0, T] \times \overline{\Omega}; R^3), \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where

$$\mathcal{R}\left(n_e, \mathbf{u}, r, \mathbf{U}\right) \equiv \int_\Omega m_e n_e \Big(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \tag{2.7}$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x + \int_\Omega \frac{m_e}{\tau} n_e \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x - \int_\Omega n_e \nabla_x \Phi \cdot \mathbf{U} \ \mathrm{d}x$$

$$+ \int_\Omega \Big((r - n_e) \partial_t P(r) + \nabla_x P(r) \cdot (r \mathbf{U} - n_e \mathbf{u}) - \mathrm{div}_x \mathbf{U} \Big(n_e \Big(P(n_e) - P(r) \Big) - E(n_e, r) \Big) \Big) \ \mathrm{d}x,$$

with

$$E(n_e, r) \equiv H(n_e) - H'(r)(n_e - r) - H(r).$$

and

$$P \equiv H', \ H(n) \equiv n \int_1^n \frac{p(s)}{s^2} \mathrm{d}s.$$

It can be deduced from (2.4) that

$$\int_\Omega \nabla_x \Phi(t, \cdot) \nabla_x \varphi \ \mathrm{d}x = \int_\Omega (N_i - n_e)(t, \cdot) \varphi \ \mathrm{d}x \text{ for any } t \in (0, T) \text{ and all test functions } \varphi \in C_c^\infty(\overline{\Omega}), \tag{2.8}$$

meaning Φ is a (strong) solution of Poisson equation (1.3). In particular, by virtue of the standard (local) elliptic regularity,

$$\Phi(t, \cdot) \in W^{2,5/3}(K) \text{ for any compact } K \subset \overline{\Omega}. \tag{2.9}$$

The *existence* of global-in-time suitable weak solutions to the compressible Navier-Stokes system in a bounded spatial domain and the no-slip boundary conditions was proved in [15, Theorem 3.1] with the help of the approximation scheme introduced in [13]. Adaptation of the method to the present problem requires only straightforward modifications. The main advantage of working directly with suitable weak solutions is that the relative entropy inequality (2.6) already implicitly includes the stability estimates necessary to perform the inviscid limit.

3 Main results

We start by introducing the scaled system. To simplify notation, we set $m_e = \varepsilon^2$, $n_e = n_\varepsilon$, $\mathbf{u} = \mathbf{u}_\varepsilon$, $\Phi = \Phi_\varepsilon$, and $N_i = \overline{n}$ - a positive constant. Accordingly, system of equations (1.1 - 1.3) reads

$$\partial_t n_\varepsilon + \mathrm{div}_x(n_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{3.1}$$

$$\partial_t(n_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(n_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x p(n_\varepsilon) = \mathrm{div}_x \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2} n_\varepsilon \nabla_x \Phi_\varepsilon - \frac{1}{\tau} n_\varepsilon \mathbf{u}_\varepsilon, \tag{3.2}$$

$$\Delta \Phi_\varepsilon = n_\varepsilon - \overline{n}, \tag{3.3}$$

with the viscous stress

$$\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}_\varepsilon) = \mu_\varepsilon \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big),\ \mu_\varepsilon > 0. \tag{3.4}$$

System (3.1 - 3.3) is supplemented with the boundary conditions

$$\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}_\varepsilon)\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{3.5}$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.6}$$

and

$$\mathbf{u}_\varepsilon(x) \to 0,\ \Phi_\varepsilon(x) \to 0 \text{ as } |x| \to \infty. \tag{3.7}$$

3.1 Ill-prepared initial data

Taking

$$r \equiv \overline{n},\ \mathbf{U} \equiv 0$$

as test functions in the relative entropy inequality (2.6) we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2} n_\varepsilon |\mathbf{u}_\varepsilon|^2 + \left[\frac{H(n_\varepsilon) - H'(\overline{n})(n_\varepsilon - \overline{n}) - H(\overline{n})}{\varepsilon^2} \right] + \frac{1}{2\varepsilon^2} |\nabla_x \Phi_\varepsilon|^2 \right)(s,\cdot)\ \mathrm{d}x \\ &+ \int_0^s \int_\Omega \left(\mu_\varepsilon \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 + \frac{1}{\tau} n_\varepsilon |\mathbf{u}_\varepsilon|^2 \right)\ \mathrm{d}x\ \mathrm{d}t \\ &\leq \int_\Omega \left(\frac{1}{2} n_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} E(n_{0,\varepsilon}, \overline{n}) + \frac{1}{2\varepsilon^2} |\nabla_x \Phi_{0,\varepsilon}|^2 \right)\ \mathrm{d}x, \end{aligned} \tag{3.8}$$

where

$$n_\varepsilon = n_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon} \tag{3.9}$$

and

$$\Phi_{0,\varepsilon} = \Delta_N^{-1}[n_{0,\varepsilon} - \overline{n}] \tag{3.10}$$

are the initial data.

Consequently, the initial data must be chosen in such a way that the expression on the right-hand side of (3.8) remains bounded for $\varepsilon \to 0$. Accordingly, we suppose that

$$n_\varepsilon(0,\cdot) = n_{0,\varepsilon} = \overline{n} + \varepsilon N_{0,\varepsilon},\ \{N_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega), \tag{3.11}$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3). \tag{3.12}$$

Moreover, the functions $N_{0,\varepsilon}$ must be taken so that $\Phi_{0,\varepsilon} = \varepsilon \Delta_N^{-1}[N_{0,\varepsilon}]$ satisfy

$$\{\nabla_x \Phi_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3). \tag{3.13}$$

3.2 Asymptotic limit for positive viscosity coefficients

Our first result concerns the asymptotic limit in the case $\mu_\varepsilon = \mu > 0$.

Theorem 3.1 *Let $\Omega \subset R^3$ be an (unbounded) C^3-domain of type (α, β, K) specified in Section 1.3 and such that the point spectrum of the Neumann Laplacian Δ_N in $L^2(\Omega)$ is empty. Suppose that the viscosity coefficient $\mu_\varepsilon = \mu > 0$ is independent of ε and that the pressure p satisfies (2.1). Let $\{n_\varepsilon, \mathbf{u}_\varepsilon, \Phi_\varepsilon\}_{\varepsilon>0}$ be a sequence of suitable weak solutions to the scaled system (3.1 - 3.7), emanating from the initial data satisfying (3.11 - 3.13).*

Then

$$\operatorname{ess} \sup_{t\in(0,T)} \|n_\varepsilon - \overline{n}\|_{L^2+L^{5/3}(\Omega)} \leq \varepsilon c, \tag{3.14}$$

and, at least for a suitable subsequence,

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \ \textit{weakly in } L^2(\Omega; R^3),$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)) \ \textit{and (strongly) in } L^2((0,T) \times K; R^3) \tag{3.15}$$

for any compact $K \subset \Omega$, where $\mathbf{U}$ is a weak solution to the incompressible (damped) Navier-Stokes system in $(0,T)\times\Omega$,

$$\operatorname{div}_x \mathbf{U} = 0, \tag{3.16}$$

$$\overline{n}\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}\Big) + \nabla_x \Pi = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \frac{\overline{n}}{\tau}\mathbf{U}, \tag{3.17}$$

with

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\nabla_x \mathbf{U})\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{3.18}$$

and

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{U}_0]. \tag{3.19}$$

Remark 3.1 *Momentum equation (3.17), together with the slip boundary conditions (3.18) and the initial condition (3.19), are understood in the weak sense, specifically, the integral identity*

$$\int_0^T \int_\Omega \overline{n}\Big(\mathbf{U}\cdot\partial_t\varphi + (\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega \Big(\mathbb{S}(\nabla_x\mathbf{U}) : \nabla_x\varphi + \frac{\overline{n}}{\tau}\mathbf{U}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \mathbf{U}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x \tag{3.20}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega}; R^3)$, $\operatorname{div}_x\varphi = 0$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$.

3.3 Inviscid limit

Our second result concerns the case of vanishing viscosity coefficient $\mu_\varepsilon \searrow 0$. In this case, the limit velocity field is expected to satisfy the incompressible Euler system (1.8 - 1.10). As is well-known, this system possesses a local-in-time solution

$$\mathbf{v} \in C([0,T_{\max}); W^{k,2}(\Omega)),\ \nabla_x\Pi \in C([0,T_{\max}); W^{k-1,2}(\Omega)) \tag{3.21}$$

provided

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 \in W^{k,2}(\Omega),\ k > 5/2,\ \operatorname{div}_x\mathbf{v}_0 = 0,\ \mathbf{v}_0\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

and provided $\Omega = R^3$, Ω is a half-space, or Ω is an (exterior) domain with compact boundary. The life-span $T_{\max}$ depends solely on $\|\mathbf{v}_0\|_{W^{k,2}(\Omega;R^3)}$, see Alazard [1], Isozaki [17], Secchi [26], among others. As a matter of fact, the damping term $\frac{1}{\tau}\mathbf{v}$ in (1.9) may extend the life-span of regular solutions, in particular if the initial data are small in comparison with $1/\tau$. In a very interesting recent paper, Chae [5] showed that a smooth solution of (1.8 - 1.10) exists *globally* in time provided $\Omega = R^3$, and $\tau < T^E_{\max}$, where $T^E_{\max}$ is the life span of the regular solution of the undamped Euler system emanating from the same initial data.

Theorem 3.2 *In addition to hypotheses of Theorem 3.1, assume that $\Omega \subset R^3$ is an infinite waveguide specified in Section 1.3.2. Moreover, we suppose that*

$$\mu_\varepsilon \searrow 0,$$

and that the initial data satisfy

$$N_{0,\varepsilon} \to N_0 \ \textit{(strongly) in } L^2(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \textit{(strongly) in } L^2(\Omega;R^3))$$

as $\varepsilon \to 0$, where

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] \in \ W^{k,2}(\Omega;R^3), \ k > 5/2.$$

Moreover, suppose that the damped Euler system (1.8- 1.10), with the initial datum $\mathbf{v}_0$, possesses a regular solution $\mathbf{v}$ defined on a time interval $[0,T_{\max})$ satisfying (3.21).

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|n_\varepsilon - \overline{n}\|_{L^2+L^{5/3}(\Omega)} \le \varepsilon c, \tag{3.22}$$

and

$$\operatorname{ess}\sup_{t\in(0,T_{\rm loc})} \left\| \mathbf{H}\left[\sqrt{\frac{n_\varepsilon}{\overline{n}}}\mathbf{u}_\varepsilon\right] - \mathbf{v} \right\|_{L^2(\Omega;R^3)} \to 0,$$

$$\sqrt{\frac{n_\varepsilon}{\overline{n}}}\mathbf{u}_\varepsilon \to \mathbf{v} \ \textit{in } L^q(0,T_{\rm loc};L^2(K;R^3)) \textit{ for any compact } K \subset \Omega, \quad 1 \le q < \infty,$$

for any $T_{\rm loc} < T_{\max}$, $T_{\rm loc} \le T$.

Remark 3.2 *The proof of Theorem 3.2 leans essentially on the $L^1 - L^\infty$ bounds for acoustic waves established in Section 7.1. Thus the conclusion of Theorem 3.2 remains valid as soon as these bounds are available. Note that Isozaki [17] established similar estimates on* exterior *domains in R^3.*

The rest of the paper is devoted to the proof of Theorems 3.1, 3.2.

4 Uniform bounds

For the ill-prepared initial data, all desired uniform bounds follow from the energy inequality (3.8). Introducing the *essential* and *residual* parts of a quantity h,

$$h = [h]_{\rm ess} + [h]_{\rm res}$$

$$[h]_{\rm ess} = \chi(n_\varepsilon)h, \ [h]_{\rm res} = h - [h_{\rm ess}],$$

where

$$\chi \in C_c^\infty(0,\infty),\ 1 \le \chi \le 1,\ \chi(n) = 1 \text{ for } n \text{ belonging to an open neighborhood of } \overline{n} \text{ in } (0,\infty),$$

we get the following list of estimates

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{n_\varepsilon}\mathbf{u}_\varepsilon(t,\cdot)\|_{L^2(\Omega;R^3)} \le c, \tag{4.1}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\nabla_x \Phi_\varepsilon(t,\cdot)\|_{L^2(\Omega;R^3)} \le \varepsilon c, \tag{4.2}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{n_\varepsilon - \overline{n}}{\varepsilon} \right]_{\text{ess}} (t,\cdot) \right\|_{L^2(\Omega)} \le c,\ \operatorname{ess}\sup_{t\in(0,T)} \int_\Omega [n_\varepsilon]^{5/3}_{\text{res}}(t,\cdot)\ \mathrm{d}x \le \varepsilon^2 c, \tag{4.3}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega [1]_{\text{res}}(t,\cdot)\ \mathrm{d}x \le \varepsilon^2 c, \tag{4.4}$$

and

$$\mu_\varepsilon \int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x \mathbf{u}_\varepsilon^t - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2\ \mathrm{d}x\ \mathrm{d}t \le c, \tag{4.5}$$

where all generic constants are independent of ε.

Estimates (4.3 - 4.5) can be combined to deduce a bound on the velocity field in the Sobolev space $L^2(0,T;W^{1,2}(\Omega;R^3))$ that is relevant in the proof of Theorem 3.1. To this end, we report the following version of *Korn's inequality* that may be of independent interest.

Proposition 4.1 *Let $\Omega \subset R^3$ be a C^2-uniform domain of type (α,β,K) introduced in Section 1.3.*

Then there exists $\delta > 0$, depending solely on the parameters (α,β,K), such that

$$\|\mathbf{w}\|^2_{W^{1,2}(\Omega;R^3)} \le c(\alpha,\beta,\delta,K) \left(\left\| \nabla_x \mathbf{w} + \nabla_x^t \mathbf{w} - \frac{2}{3}\mathrm{div}_x \mathbf{w}\mathbb{I} \right\|^2_{L^2(\Omega;R^{3\times3})} + \int_{\Omega\setminus V} |\mathbf{w}|^2\ \mathrm{d}x \right) \tag{4.6}$$

for any measurable set V, $|V| < \delta$, and for all $\mathbf{w} \in W^{1,2}(\Omega;R^3)$.

Proof:

In view of the standard decomposition technique and partition of unity, it is enough to show (4.6) on each set

$$U^-_{\alpha,\beta,K} = \{(y,x_3)\ |h(y) - \beta < x_3 < h(y),\ |y| < \alpha\}.$$

Revoking the result [4, Proposition 4.1], we have

$$\|\mathbf{w}\|^2_{W^{1,2}(U^-;R^3)} \le c(\alpha,\beta,K) \left(\left\| \nabla_x \mathbf{w} + \nabla_x^t \mathbf{w} - \frac{2}{3}\mathrm{div}_x \mathbf{w}\mathbb{I} \right\|^2_{L^2(U^-;R^{3\times3})} + \int_{U^-} |\mathbf{w}|^2\ \mathrm{d}x \right). \tag{4.7}$$

As a matter of fact, the constant c in (4.7) depends only the Lipschitz constant of the function h and width of U^- given in terms of α, β.

Furthermore, we have

$$|U^-_{\alpha,\beta,K}| \ge 2\delta > 0$$

for a certain $\delta(\alpha, \beta, K) > 0$. In particular,

$$|U^- \setminus V| > \delta \text{ for any measurable set } V, \ |V| < \delta.$$

Now, arguing by contradiction, we construct sequences

$$\{h_n\}_{n=1}^{\infty}, \ \|h_n\|_{C^2(R^2)} \leq K, \ U_n^- = \left\{ (y, x_3) \ \Big| \ h_n(y) - \beta < x_3 < h_n(y), \ |y| < \alpha \right\},$$

$$h_n \to h \text{ in } C^1(\{|y| \leq \alpha),$$

$$\{V_n\}_{n=1}^{\infty}, \ |V_n| < \delta,$$

and

$$\{\mathbf{w}_n\}_{n=1}^{\infty}, \ \|\mathbf{w}_n\|_{W^{1,2}(U_n^-;R^3)} = 1,$$

such that

$$\left(\left\| \nabla_x \mathbf{w}_n + \nabla_x^t \mathbf{w}_n - \frac{2}{3} \mathrm{div}_x \mathbf{w}_n \mathbb{I} \right\|^2_{L^2(U_n^-;R^{3\times3})} + \int_{U_n^- \setminus V_n} |\mathbf{w}_n|^2 \ \mathrm{d}x \right) \to 0 \text{ as } n \to \infty.$$

Because the domains U_n^- are uniformly Lipschitz, we can extend $\mathbf{w}_n$ as $\tilde{\mathbf{w}}_n$ on the cylinder

$$U = \{(y, x_3) \mid |x_3| < 2 + \beta + \alpha K, \ |y| < \alpha\}$$

in such a way that

$$\|\tilde{\mathbf{w}}_n\|_{W^{1,2}(U;R^3)} \leq c(\alpha, \beta, K) \|\mathbf{w}_n\|_{W^{1,2}(U_n^-;R^3)}.$$

Since $W^{1,2}(U; R^3)$ is compactly embedded into $L^2(U; R^3)$, we may use (4.7) to deduce that

$$\tilde{\mathbf{w}}_n \to \mathbf{w} \text{ in } W^{1,2}(U; R^3),$$

where

$$\nabla_x \mathbf{w} + \nabla_x^t \mathbf{w} - \frac{2}{3} \mathrm{div}_x \mathbf{w} \mathbb{I} = 0, \ \mathbf{w} \not\equiv 0 \text{ in the set } \left\{ (y, x_3) \ \Big| \ h(y) - \beta < x_3 < h(y), \ |y| < \alpha \right\}. \tag{4.8}$$

On the other hand,

$$\int_U 1_{U_n^- \setminus V_n} |\mathbf{w}_n|^2 \ \mathrm{d}x \to \int_U \chi |\mathbf{w}|^2 \ \mathrm{d}x = 0 \tag{4.9}$$

where

$$1_{U_n^- \setminus V_n} \to \chi \text{ weakly-(*) in } L^\infty(\Omega), \ \chi \geq 0, \ \int_U \chi \ \mathrm{d}x > 0.$$

However, relation (4.8) implies that $\mathbf{w}$ is a (nonzero) conformal Killing vector (see Reshetnyak [23]) vanishing, by virtue (4.9), on a set of positive measure, which is impossible.

Q.E.D.

Thus, finally, taking V the "residual set", $V = \mathrm{supp}[1]_{\mathrm{res}}$ we may combine the estimates (3.8), (4.4), and (4.5) with Proposition 4.1 to conclude that

$$\mu_\varepsilon \int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega;R^3)} \ \mathrm{d}t \leq c. \tag{4.10}$$

5 Acoustic equation

As already pointed out, the essential piece of information necessary to carry out the asymptotic limit is contained in the oscillatory component of the velocity field responsible for propagation of acoustic waves. Introducing new variables

$$N_\varepsilon = \frac{n_\varepsilon - \overline{n}}{\varepsilon},\ \mathbf{V}_\varepsilon = n_\varepsilon \mathbf{u}_\varepsilon$$

we can formally rewrite system (3.1), (3.2) in the form

$$\varepsilon \partial_t N_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0, \tag{5.1}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + p'(\overline{n}) \nabla_x N_\varepsilon + \frac{\overline{n}}{\varepsilon} \nabla_x \Phi_\varepsilon \tag{5.2}$$

$$= \varepsilon \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varepsilon \mathrm{div}_x (n_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{\varepsilon}{\tau} n_\varepsilon \mathbf{u}_\varepsilon - N_\varepsilon \nabla_x \Phi_\varepsilon - \frac{1}{\varepsilon} \nabla_x \Big(p(n_\varepsilon) - p'(\overline{n})(n_\varepsilon - \overline{n}) - p(\overline{n}) \Big),$$

supplemented with the boundary condition

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

System (5.1), (5.2) is usually called *acoustic equation*, see Lighthill [21]. Its (rigorous) weak formulation reads

$$\int_0^T \int_\Omega \Big(\varepsilon N_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t = -\varepsilon \int_\Omega N_{0,\varepsilon} \varphi(0,\cdot)\ \mathrm{d}x \tag{5.3}$$

for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$, and

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \nabla_x \varphi + p'(\overline{n}) N_\varepsilon \Delta\varphi - \overline{n} N_\varepsilon \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{5.4}$$

$$= \varepsilon \int_0^T \int_\Omega \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x^2 \varphi\ \mathrm{d}x\ \mathrm{d}t - \varepsilon \int_0^T \int_\Omega \left(n_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \frac{p(n_\varepsilon) - p'(\overline{n})(n_\varepsilon - \overline{n}) - p(\overline{n})}{\varepsilon^2} \mathbb{I} \right) : \nabla_x^2 \varphi\ \mathrm{d}x\ \mathrm{d}t$$

$$+ \varepsilon \int_0^T \int_\Omega \left(\frac{1}{\tau} n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi + N_\varepsilon \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \cdot \nabla_x \varphi \right)\ \mathrm{d}x\ \mathrm{d}t - \varepsilon \int_\Omega n_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \nabla_x \varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C^\infty_c(\Omega)$, $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Moreover, we rewrite

$$\int_\Omega N_\varepsilon \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \cdot \nabla_x \varphi\ \mathrm{d}x = \frac{1}{\varepsilon^2} \int_\Omega \Delta\Phi_\varepsilon \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi\ \mathrm{d}x$$

$$= \frac{1}{\varepsilon^2} \int_\Omega \left(\frac{1}{2} \nabla_x |\nabla_x \Phi_\varepsilon|^2 \cdot \nabla_x \varphi + \mathrm{div}_x (\nabla_x \Phi_\varepsilon \otimes \nabla_x \Phi_\varepsilon) \cdot \nabla_x \varphi \right)\ \mathrm{d}x$$

$$= -\frac{1}{\varepsilon^2} \int_\Omega \left(\frac{1}{2} |\nabla_x \Phi_\varepsilon|^2 \cdot \Delta\varphi + (\nabla_x \Phi_\varepsilon \otimes \nabla_x \Phi_\varepsilon) : \nabla_x^2 \varphi \right)\ \mathrm{d}x.$$

Furthermore, it follows directly from the uniform bounds established in (3.8 - 4.5) that (5.4) can be written as

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \nabla_x \varphi + p'(\overline{n}) N_\varepsilon \Delta\varphi - \overline{n} N_\varepsilon \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{5.5}$$

$$= \varepsilon \int_0^T \int_\Omega \Big(\mathbb{G}^1_\varepsilon : \nabla_x^2 \varphi + \mathbb{G}^2_\varepsilon : \nabla_x^2 \varphi + \frac{1}{\tau} n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \varepsilon \int_\Omega n_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \nabla_x \varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in C^\infty_c(\Omega)$, $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where

$$\{\mathbb{G}^1_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega;R^{3\times 3})), \tag{5.6}$$

and

$$\{\mathbb{G}^2_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega;R^{3\times 3})). \tag{5.7}$$

5.1 Neumann Laplacian

At this stage, it is convenient to rewrite the acoustic system (5.3), (5.5) in terms of a single self-adjoint operator $\mathcal{A}$ in $L^2(\Omega)$, specifically,

$$\mathcal{A} = -p'(\overline{n})\Delta_N + \overline{n}\mathbb{I},$$

with

$$\mathcal{D}(\mathcal{A}) = \Big\{ w \in W^{1,2}(\Omega) \ \Big| $$

$$\int_\Omega \left(p'(\overline{n}) \nabla_x w \cdot \nabla_x \varphi + \overline{n} w \varphi \right) \ \mathrm{d}x = \int_\Omega g\varphi \ \mathrm{d}x \text{ for any } \varphi \in C^\infty_c(\overline{\Omega}) \text{ for a certain } g \in L^2(\Omega) \Big\}.$$

Given the regularity of the boundary $\partial\Omega$, it can be shown that

$$\mathcal{D}(\mathcal{A}) = \Big\{ w \in W^{2,2}(\Omega) \ \Big| \ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0 \Big\}, \ \mathcal{A}[w] = -p'(\overline{n})\Delta w + \overline{n} w. \tag{5.8}$$

Furthermore, since Ω is of uniform C^3-class, the classical elliptic theory yields

$$\mathcal{D}(\mathcal{A}^2) \subset C^{2+\nu} \cap W^{2,\infty}(\overline{\Omega}) \tag{5.9}$$

for a certain $\nu > 0$. We remark that all we need is only uniform $C^{2+\nu}-$regularity of the boundary instead of C^3.

5.2 Acoustic equation - abstract formulation

In view of (5.8), (5.9), and the uniform bounds established in (5.6), (5.7), the acoustic equation (5.3), (5.5) can be written in a concise form:

$$\int_0^T \int_\Omega \Big(\varepsilon N_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = -\varepsilon \int_\Omega N_{0,\varepsilon} \varphi(0,\cdot) \ \mathrm{d}x \tag{5.10}$$

for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$, and

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \nabla_x \varphi - N_\varepsilon \mathcal{A}[\varphi] \Big) \ \mathrm{d}x \ \mathrm{d}t = \varepsilon \int_0^T \int_\Omega F_\varepsilon \mathcal{A}^2[\varphi] \ \mathrm{d}x \ \mathrm{d}t - \varepsilon \int_\Omega n_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \nabla_x \varphi(0,\cdot) \ \mathrm{d}x \tag{5.11}$$

for any $\varphi \in C^1([0,T);\mathcal{D}(\mathcal{A}^2))$, with

$$\{F_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega)). \tag{5.12}$$

Thus, using the standard variation-of-constants formula, we obtain

$$N_\varepsilon = \frac{1}{2}\left(\exp\left(\mathrm{i}\sqrt{\mathcal{A}}\frac{t}{\varepsilon}\right)\left[N_{0,\varepsilon} + \frac{\mathrm{i}}{\sqrt{\mathcal{A}}}Z_{0,\varepsilon}\right] + \exp\left(-\mathrm{i}\sqrt{\mathcal{A}}\frac{t}{\varepsilon}\right)\left[N_{0,\varepsilon} - \frac{\mathrm{i}}{\sqrt{\mathcal{A}}}Z_{0,\varepsilon}\right]\right) \tag{5.13}$$

$$+\frac{1}{2}\int_0^t \left(\exp\left(\mathrm{i}\sqrt{\mathcal{A}}\frac{t-s}{\varepsilon}\right) - \exp\left(-\mathrm{i}\sqrt{\mathcal{A}}\frac{t-s}{\varepsilon}\right)\right)\left[\frac{\mathrm{i}}{\sqrt{\mathcal{A}}}\mathcal{A}^2[F_\varepsilon]\right]\ \mathrm{d}s,$$

$$Z_\varepsilon = \frac{1}{2}\left(\exp\left(\mathrm{i}\sqrt{\mathcal{A}}\frac{t}{\varepsilon}\right)\left[Z_{0,\varepsilon} - \mathrm{i}\sqrt{\mathcal{A}}[N_{0,\varepsilon}]\right] + \exp\left(-\mathrm{i}\sqrt{\mathcal{A}}\frac{t}{\varepsilon}\right)\left[Z_{0,\varepsilon} + \mathrm{i}\sqrt{\mathcal{A}}[N_{0,\varepsilon}]\right]\right) \tag{5.14}$$

$$+\frac{1}{2}\int_0^t \left(\exp\left(\mathrm{i}\sqrt{\mathcal{A}}\frac{t-s}{\varepsilon}\right) + \exp\left(-\mathrm{i}\sqrt{\mathcal{A}}\frac{t-s}{\varepsilon}\right)\right)\left[\mathcal{A}^2[F_\varepsilon]\right]\ \mathrm{d}s,$$

where Z_ε is interpreted as

$$\int_\Omega Z_\varepsilon G(\mathcal{A})[\varphi]\ \mathrm{d}x = -\int_\Omega \mathbf{V}_\varepsilon \cdot \nabla_x (G(\mathcal{A})[\varphi])\ \mathrm{d}x \text{ for any } G \in C_c^\infty(\overline{n}, \infty),\ \varphi \in C_c^\infty(\Omega), \tag{5.15}$$

in particular,

$$\int_\Omega Z_{0,\varepsilon} G(\mathcal{A})[\varphi]\ \mathrm{d}x = -\int_\Omega n_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \cdot \nabla_x (G(\mathcal{A})[\varphi])\ \mathrm{d}x \text{ for any } G \in C_c^\infty(\overline{n}, \infty),\ \varphi \in C_c^\infty(\Omega). \tag{5.16}$$

Note that the spectrum of the operator $\mathcal{A}$ is the half-line $[\overline{n}, \infty)$.

5.3 Application of RAGE theorem

With the explicit formulas (5.13), (5.14) at hand, we are ready to show local energy decay for N_ε and the acoustic waves represented by the gradient component $\mathbf{H}^\perp[\mathbf{V}_\varepsilon]$. To this end, we employ the following version of the celebrated RAGE theorem, see Cycon et al. [7, Theorem 5.8]:

Theorem 5.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto the space of continuity H_c of A, specifically,*

$$H = H_c \oplus \mathrm{cl}_H\Big\{\mathrm{span}\{w \in H \mid w \text{ an eigenvector of } A\}\Big\}.$$

Then

$$\left\| \frac{1}{\tau}\int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA)\ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \text{ as } \tau \to \infty. \tag{5.17}$$

We apply Theorem 5.1 to $H = L^2(\Omega)$, $A = -\sqrt{\mathcal{A}}$, $C = \chi^2 G(\mathcal{A})$, with $\chi \in C_c^\infty(\Omega)$, $\chi \geq 0$. In accordance with hypotheses of Theorem 3.1, the point spectrum of $\mathcal{A}$ is empty, and we deduce that

$$\int_0^T \left\| \chi G(\mathcal{A}) \exp\left(\mathrm{i}\sqrt{\mathcal{A}}\frac{t}{\varepsilon}\right)[X] \right\|^2_{L^2(\Omega)}\ \mathrm{d}t \leq \omega(\varepsilon)\|X\|^2_{L^2(\Omega)}, \tag{5.18}$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, going back to (5.13), (5.14) we may infer that

$$\left\{ t \mapsto \int_\Omega N_\varepsilon(t, \cdot) G(\mathcal{A})[\varphi]\ \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T),$$

and, similarly,

$$\left\{ t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \nabla_x (G(\mathcal{A})[\varphi]) \ \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T)$$

as $\varepsilon \to \infty$ for any $G \in C_c^\infty(\overline{n}, \infty)$, $\varphi \in C_c^\infty(\Omega)$. Thus, by means of density argument,

$$N_\varepsilon \to 0 \text{ in } L^q(0,T; (L^2 + L^{5/3})_{\mathrm{weak}-(*)}(\Omega)) \text{ for any } 1 \leq q < \infty, \tag{5.19}$$

while

$$\left\{ t \mapsto \int_\Omega n_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{H}^\perp[\varphi] \ \mathrm{d}x \right\} \to 0 \text{ in } L^q(0,T) \text{ for any } 1 \leq q < \infty, \ \varphi \in C_c^\infty(\Omega). \tag{5.20}$$

6 Compactness of the solenoidal part - proof of Theorem 3.1

In this section, we complete the proof of Theorem 3.1. To begin, we remark that relation (3.14) follows directly from (4.3), while (4.10) implies that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{6.1}$$

at least for a subsequence as the case may be. Moreover, the vector field $\mathbf{U}$ is solenoidal and satisfies the impermeability condition $\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Next, the uniform bounds (4.2), (4.3), together with the standard elliptic theory, yield

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \nabla_x \left(\frac{\Phi_\varepsilon(t, \cdot)}{\varepsilon} \right) \right\|_{W^{1,5/3}(K, R^3)} \leq c(K) \text{ for any compact } K \subset \overline{\Omega}, \tag{6.2}$$

which, combined with (5.19), yields

$$\int_0^T \int_\Omega \frac{n_\varepsilon - \overline{n}}{\varepsilon} \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0 \text{ for any } \varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3). \tag{6.3}$$

Taking $\varphi = C_c^\infty((0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$, as a test function in the momentum equation(2.3) and making use of (6.3), we deduce that

$$\{\mathbf{H}[n_\varepsilon \mathbf{u}_\varepsilon]\}_{\varepsilon > 0} \text{ is precompact in } C_{\mathrm{weak}-(*)}([0,T]; L^2 + L^{5/4}(\Omega; R^3)). \tag{6.4}$$

Indeed the only quantity term reads

$$\frac{1}{\varepsilon^2} n_\varepsilon \nabla_x \Phi_\varepsilon = \frac{\overline{n}}{\varepsilon^2} \nabla_x \Phi_\varepsilon + \frac{n_\varepsilon - \overline{n}}{\varepsilon} \nabla_x \frac{\Phi_\varepsilon}{\varepsilon},$$

where the former term is a gradient, while the latter satisfies (6.3).

Putting together (5.20), (6.1), (6.4), with (3.14), we deduce the desired conclusion

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ (strongly) in } L^2((0,T) \times K; R^3) \text{ for any compact } K \subset \Omega. \tag{6.5}$$

With relations (6.3), (6.5) at hand, it is not difficult to perform the limit $\varepsilon \to 0$ in the weak formulation of momentum equation (3.2) to obtain (3.20).

We have proved Theorem 3.1.

7 Zero viscosity limit - proof of Theorem 3.2

Our ultimate goal is to prove Theorem 3.2. The basic tool here is the relative entropy inequality (2.6) satisfied by the suitable weak solutions. Taking $n_e = n_\varepsilon$, $\mathbf{u} = \mathbf{u}_\varepsilon$, $r = \overline{n}$ in the rescaled variant of (2.6) we obtain

$$\int_\Omega \left(\frac{1}{2} n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}|^2 + \frac{1}{\varepsilon^2} E(n_\varepsilon, \overline{n}) + \frac{1}{2} \left| \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \right|^2 \right) (s, \cdot) \, \mathrm{d}x \tag{7.1}$$

$$+ \int_0^s \int_\Omega [\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}_\varepsilon) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u}_\varepsilon - \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega \frac{1}{\tau} n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} n_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{U}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} E(n_{0,\varepsilon}, \overline{n}) + \frac{1}{2} \left| \nabla_x \left(\frac{\Phi_{0,\varepsilon}}{\varepsilon} \right) \right|^2 \right) \, \mathrm{d}x + \int_0^s \mathcal{R}\left(n_\varepsilon, \mathbf{u}_\varepsilon, \overline{n}, \mathbf{U}\right) \, \mathrm{d}t,$$

with

$$\mathcal{R}\left(n_e, \mathbf{u}_\varepsilon, \overline{n}, \mathbf{U}\right) \equiv \int_\Omega n_\varepsilon \Big(\partial_t \mathbf{U} + \mathbf{u}_\varepsilon \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}_\varepsilon) \, \mathrm{d}x \tag{7.2}$$

$$+ \int_\Omega \mathbb{S}_\varepsilon(\nabla_x \mathbf{U}) : \nabla_x(\mathbf{U} - \mathbf{u}_\varepsilon) \, \mathrm{d}x + \int_\Omega \frac{1}{\tau} n_\varepsilon \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}_\varepsilon) \, \mathrm{d}x - \frac{1}{\varepsilon^2} \int_\Omega n_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U} \, \mathrm{d}x$$

$$-\frac{1}{\varepsilon^2} \int_\Omega \mathrm{div}_x \mathbf{U} \Big(n_\varepsilon \Big(P(n_\varepsilon) - P(\overline{n}) \Big) - E(n_\varepsilon, \overline{n}) \Big) \, \mathrm{d}x.$$

Furthermore, we take

$$\mathbf{U} = \mathbf{U}_{\varepsilon,\delta} = \mathbf{v} + \nabla_x \Psi_{\varepsilon,\delta},$$

where $\mathbf{v}$ is the (unique) solution of the damped Euler system (1.8-1.10), emanating from the initial data $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$, and $\nabla_x \Psi_{\varepsilon,\delta}$ mimicks the oscillatory part of the velocity field. Specifically, we take

$$\varepsilon \partial_t s_{\varepsilon,\delta} + \Delta \Psi_{\varepsilon,\delta} = 0, \tag{7.3}$$

$$\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} + p'(\overline{n}) \nabla_x s_{\varepsilon,\delta} - \overline{n} \nabla_x \Delta_N^{-1} s_{\varepsilon,\delta} + \frac{\varepsilon}{\tau} \nabla_x \Psi_{\varepsilon,\delta} = 0, \ \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{7.4}$$

which is nothing other than a slightly modified homogeneous part of the acoustic system (5.10), (5.11). The initial data are taken in the form

$$s_{\varepsilon,\delta}(0, \cdot) = \frac{1}{\overline{n}} [N_{0,\varepsilon}]_\delta, \ \Psi_{\varepsilon,\delta}(0, \cdot) = [\Psi_{0,\varepsilon}]_\delta, \text{ with } \nabla_x \Psi_{0,\varepsilon} = \mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}], \tag{7.5}$$

where the brackets $[\cdot]_\delta$ denote a suitable regularization operator specified in Section 7.1 below.

Keeping (7.3 - 7.4) in mind, we can rewrite the remainder (7.2) in the form

$$\mathcal{R}\left(n_e, \mathbf{u}_\varepsilon, \overline{n}, \mathbf{U}_{\varepsilon,\delta}\right) \tag{7.6}$$

$$= \int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{u}_\varepsilon \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \, \mathrm{d}x + \int_\Omega \big(n_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} + n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x^2 \Psi_{\varepsilon,\delta} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \big) \, \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}_\varepsilon(\nabla_x \mathbf{U}_{\varepsilon,\delta}) : \nabla_x(\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \, \mathrm{d}x - \int_\Omega \Delta \Psi_{\varepsilon,\delta} \left(\frac{P(n_\varepsilon) - P(\overline{n})}{\varepsilon} \frac{n_\varepsilon - \overline{n}}{\varepsilon} - \frac{E(n_\varepsilon, \overline{n})}{\varepsilon^2} \right) \, \mathrm{d}x$$

$$\int_\Omega \left(-n_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Psi_{\varepsilon,\delta} + \frac{1}{2} n_\varepsilon \partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2 - \frac{\overline{n}}{\varepsilon} \Delta \Psi_{\varepsilon,\delta} \frac{P(n_\varepsilon) - P(\overline{n})}{\varepsilon} \right) \, \mathrm{d}x$$

$$+\int_\Omega \frac{1}{\tau} n_\varepsilon \nabla_x \Psi_{\varepsilon,\delta}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x - \frac{1}{\varepsilon^2}\int_\Omega n_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x.$$

Moreover, we compute

$$\int_\Omega n_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Psi_{\varepsilon,\delta}\ \mathrm{d}x = p'(\overline{n})\int_\Omega N_\varepsilon \partial_t s_{\varepsilon,\delta}\ \mathrm{d}x - p'(\overline{n})\left[\int_\Omega N_\varepsilon s_{\varepsilon,\delta}\ \mathrm{d}x\right]_{t=0}^{t=s}$$

$$-\overline{n}\int_\Omega N_\varepsilon \partial_t \Delta_N^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x + \overline{n}\left[\int_\Omega N_\varepsilon \Delta_N^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x\right]_{t=0}^{t=s} - \frac{1}{\tau}\int_\Omega n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta}\ \mathrm{d}x$$

and

$$\int_\Omega \frac{1}{2} n_\varepsilon \partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2\ \mathrm{d}x = \int_\Omega \frac{1}{2}(n_\varepsilon - \overline{n})\partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2\ \mathrm{d}x + \int_\Omega \frac{1}{2}\overline{n}\partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2\ \mathrm{d}x,$$

where, by virtue of (7.3), (7.4),

$$\int_\Omega \frac{1}{2}\overline{n}\partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2\ \mathrm{d}x = -\int_\Omega p'(n)\frac{\overline{n}}{2}\partial_t |s_{\varepsilon,\delta}|^2\ \mathrm{d}x + \frac{\overline{n}^2}{\varepsilon}\int_\Omega \nabla_x \Psi_{\varepsilon,\delta}\cdot \nabla_x \Delta_N^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x - \frac{\overline{n}}{\tau}\int_\Omega |\nabla_x \Psi_{\varepsilon,\delta}|^2\ \mathrm{d}x.$$

Next, we have

$$\frac{\overline{n}}{\varepsilon}\int_\Omega \Delta\Psi_{\varepsilon,\delta}\frac{P(n_\varepsilon)-P(\overline{n})}{\varepsilon}\ \mathrm{d}x$$

$$=\frac{1}{\varepsilon}\int_\Omega p'(\overline{n})\Delta\Psi_{\varepsilon,\delta}N_\varepsilon\ \mathrm{d}x + \frac{\overline{n}}{\varepsilon^2}\int_\Omega \left(P(n_\varepsilon) - \frac{p'(\overline{n})}{\overline{n}}(n_\varepsilon - \overline{n}) - P(\overline{n})\right)\Delta\Psi_{\varepsilon,\delta}\ \mathrm{d}x,$$

where

$$\frac{1}{\varepsilon}\int_\Omega \Delta\Psi_{\varepsilon,\delta}N_\varepsilon\ \mathrm{d}x = -\int_\Omega \partial_t s_{\varepsilon,\delta}N_\varepsilon\ \mathrm{d}x.$$

Finally,

$$\frac{1}{\varepsilon^2}\int_\Omega n_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x = \int_\Omega N_\varepsilon \nabla_x\left(\frac{\Phi_\varepsilon}{\varepsilon}\right)\cdot \mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x + \frac{\overline{n}}{\varepsilon}\int_\Omega \nabla_x\left(\frac{\Phi_\varepsilon}{\varepsilon}\right)\cdot\nabla_x \Psi_{\varepsilon,\delta}\ \mathrm{d}x$$

$$=\int_\Omega N_\varepsilon \nabla_x\left(\frac{\Phi_\varepsilon}{\varepsilon}\right)\cdot \mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x + \overline{n}\int_\Omega N_\varepsilon \partial_t \Delta_N^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x.$$

Summing up the previous considerations we may infer that

$$\int_\Omega \left(\frac{1}{2}n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v} - \nabla_x \Psi_{\varepsilon,\delta}|^2 + \frac{1}{\varepsilon^2}E(n_\varepsilon,\overline{n}) + \frac{1}{2}\left|\nabla_x\left(\frac{\Phi_\varepsilon}{\varepsilon}\right)\right|^2\right)(s,\cdot)\ \mathrm{d}x \tag{7.7}$$

$$+\int_\Omega \left(p'(\overline{n})\frac{\overline{n}}{2}|s_{\varepsilon,\delta}|^2 - p'(\overline{n})N_\varepsilon s_{\varepsilon,\delta} + \overline{n}N_\varepsilon \Delta_N^{-1}[s_{\varepsilon,\delta}] + \frac{\overline{n}^2}{2}\left|(-\Delta_n)^{-1/2}[s_{\varepsilon,\delta}]\right|^2\right)(s,\cdot)\ \mathrm{d}x$$

$$+\frac{\mu_\varepsilon}{2}\int_0^s\int_\Omega \left|\nabla_x(\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) + \nabla_x^t(\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) - \frac{2}{3}\mathrm{div}_x(\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta})\mathbb{I}\right|^2\ \mathrm{d}x\ \mathrm{d}t + \int_0^s\int_\Omega \frac{1}{\tau}n_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_\Omega \left(+\frac{1}{2}n_{0,\varepsilon}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{H}[\mathbf{u}_0] - \nabla_x[\Psi_{0,\varepsilon}]_\delta\right|^2 + \frac{1}{\varepsilon^2}E(n_{0,\varepsilon},\overline{n}) + \frac{1}{2}\left|\nabla_x\left(\frac{\Phi_{0,\varepsilon}}{\varepsilon}\right)\right|^2\right)\ \mathrm{d}x +$$

$$+\int_\Omega \left(p'(\overline{n}) \frac{\overline{n}}{2} |[N_{0,\varepsilon}]_\delta|^2 - p'(\overline{n}) N_{0,\varepsilon} [N_{0,\varepsilon}]_\delta + \overline{n} N_{0,\varepsilon} \Delta_N^{-1} [N_{0,\varepsilon}]_\delta + \frac{\overline{n}^2}{2} \left| (-\Delta_n)^{-1/2} [N_{0,\varepsilon}]_\delta \right|^2 \right) \ \mathrm{d}x$$

$$+ \int_0^s Q_{\varepsilon,\delta} \ \mathrm{d}t,$$

where

$$Q_{\varepsilon,\delta} =$$

$$= \int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{u}_\varepsilon \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x + \int_\Omega \big(n_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} + n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x^2 \Psi_{\varepsilon,\delta} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \big) \ \mathrm{d}x$$

$$+ \frac{1}{\tau} \int_\Omega (n_\varepsilon - \overline{n}) \mathbf{U}_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x + \frac{1}{\tau} \int_\Omega n_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x + \frac{1}{2} \int_\Omega (n_\varepsilon - \overline{n}) \partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2 \ \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}_\varepsilon (\nabla_x \mathbf{U}_{\varepsilon,\delta}) : \nabla_x (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x - \int_\Omega \Delta \Psi_{\varepsilon,\delta} \left(\frac{P(n_\varepsilon) - P(\overline{n})}{\varepsilon} \frac{n_\varepsilon - \overline{n}}{\varepsilon} - \frac{E(n_\varepsilon, \overline{n})}{\varepsilon^2} \right) \ \mathrm{d}x$$

$$- \frac{\overline{n}}{\varepsilon^2} \int_\Omega \left(P(n_\varepsilon) - \frac{p'(\overline{n})}{\overline{n}} (n_\varepsilon - \overline{n}) - P(\overline{n}) \right) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x - \int_\Omega N_\varepsilon \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x.$$

7.1 Dispersive estimates of the oscillatory component

Our goal is to show that solutions $s_{\varepsilon,\delta}$, $\Psi_{\varepsilon,\delta}$ of the homogeneous "acoustic" equation (7.3), (7.4) decay to zero in the L^∞ norm as $\varepsilon \to 0$ for any positive time t. To this end, we start with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(|\nabla_x \Psi_{\varepsilon,\delta}|^2 + p'(\overline{n}) |s_{\varepsilon,\delta}|^2 + \overline{n} \left| (-\Delta_N)^{-1/2} [s_{\varepsilon,\delta}] \right|^2 \right) \ \mathrm{d}x + \frac{2}{\tau} \int_\Omega |\nabla_x \Psi_{\varepsilon,\delta}|^2 \ \mathrm{d}x = 0 \tag{7.8}$$

yielding, in particular, existence and uniqueness of (weak) solutions to problem (7.3), (7.4) provided the initial data are smooth and decay sufficiently fast for $|x| \to \infty$.

Taking advantage of the special geometry of waveguides, we consider the functions $w_k(z)$, $z \in B$ - the eigenfunctions of the Neumann Laplacian $-\Delta_{N,B}$ in the (bounded) domain $B \subset R^{3-L}$:

$$-\Delta_{N,B} w_k = \lambda_k w_k \text{ in } B, \ \nabla_z w_k \cdot \mathbf{n}_z|_{\partial B} = 0, \ \lambda_0 = 0 < \lambda_1 \le \ldots \le \lambda_k \le \ldots, \ k = 0, 1, \ldots.$$

The smoothing operators $[g]_\delta$, $g = g(x)$, $x = [y, z]$ are defined as

$$[g]_\delta(y, z) = \sum_{0 \le k < 1/\delta} \kappa_\delta(y) * \Big(\psi_\delta(y) A_k[g](y) \Big) w_k(z), \tag{7.9}$$

where

$$A_k[g](y) = \frac{1}{|B|} \int_B g(y, z) w_k(z) \ \mathrm{d}z,$$

$\psi_\delta \in C_c^\infty(R^L)$ is a cut-off function,

$$0 \le \psi_\delta \le 1, \ \psi_\delta(y) = \left\{ \begin{array}{l} 1 \text{ for } |y| < 1/\delta, \\ \\ 0 \text{ for } |y| > 2/\delta, \end{array} \right.$$

and κ_δ is a family of standard regularizing kernels in the $y-$variable.

A short inspection of (7.3), (7.4) yields

$$s_{\varepsilon,\delta}(t,x) = \exp\left(\frac{t}{2\tau}\right)\tilde{s}_{\varepsilon,\delta}\left(\frac{t}{\varepsilon},x\right), \tag{7.10}$$

where $\tilde{s}_{\varepsilon,\delta}$ is the unique solution of the Klein-Gordon equation

$$\partial^2_{t,t}\tilde{s}_{\varepsilon,\delta} - p'(\overline{n})\Delta\tilde{s}_{\varepsilon,\delta} + \left(\overline{n} - \frac{1}{4}\frac{\varepsilon^2}{\tau_2}\right)\tilde{s}_{\varepsilon,\delta} = 0,\ \nabla_x\tilde{s}_{\varepsilon,\delta}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

emanating from the initial data

$$\tilde{s}_{\varepsilon,\delta}(0,\cdot) = \frac{1}{\overline{n}}[N_{0,\varepsilon}]_\delta,\ \partial_t\tilde{s}_{\varepsilon,\delta}(0,\cdot) = -\Delta[\Psi_{0,\varepsilon}]_\delta. \tag{7.11}$$

Consequently, thanks to the specific choice of the smoothing operators (7.9), solutions $\tilde{s}_{\varepsilon,\delta}$ take the form

$$\tilde{s}_{\varepsilon,\delta}(t,x) = \sum_{0\leq k\leq 1/\delta} S_{k,\varepsilon,\delta}(t,y)w_k(z),$$

where $S_k(t,\cdot)$ solve the Klein-Gordon equation

$$\partial^2_{t,t}S_{k,\varepsilon,\delta} - p'(\overline{n})\Delta_y S_{k,\varepsilon,\delta} + \left(\overline{n} - \frac{1}{4}\frac{\varepsilon^2}{\tau^2} + p'(\overline{n})\lambda_k\right)S_{k,\varepsilon,\delta} = 0 \tag{7.12}$$

for y belonging to the "flat" space R^L, and with the initial data uniquely determined through (7.11). Thus, employing the standard $L^1 - L^\infty$ estimates for the Klein-Gordon equation 7.12 (see for instance Lesky and Racke [20, Lemma 2.4]), we have

$$\|S_{k,\varepsilon,\delta}(t,\cdot)\|_{L^\infty(R^L)} \leq \frac{c(k)}{(1+t)^{L/2}}\Big(\|S_{k,\varepsilon,\delta}(0,\cdot)\|_{W^{K,1}(R^L)} + \|\partial_t S_{k,\varepsilon,\delta}(0,\cdot)\|_{W^{K-1,1}(R^L)}\Big),\ K = \left[\frac{L+3}{2}\right].$$

Going back to (7.10) we may infer that

$$\|s_{\varepsilon,\delta}(t,\cdot)\|_{L^\infty(\Omega)} \leq \omega(t_0,\varepsilon,\delta),\ t\in[t_0,T],\ t_0>0, \tag{7.13}$$

and, using (7.3),

$$\|\Delta\Psi_{\varepsilon,\delta}(t,\cdot)\|_{L^\infty(\Omega)} \leq \omega(t_0,\varepsilon,\delta),\ t\in[t_0,T],\ t_0>0, \tag{7.14}$$

where $\omega(t_0,\varepsilon,\delta)\to 0$ if $\varepsilon\to 0$ for any fixed $t_0>0$, $\delta>0$.

Finally, we claim the standard energy bounds

$$\|\nabla_x\Psi_{\varepsilon,\delta}\|_{L^\infty(0,T;W^{m,2}(\Omega;R^3))} \leq c(m,\delta), \tag{7.15}$$

$$\|s_{\varepsilon,\delta}\|_{L^\infty(0,T;W^{m,2}(\Omega;R^3))} \leq c(m,\delta),\ m = 0,1,\ldots, \tag{7.16}$$

where the constants are independent of ε for any fixed $\delta>0$.

7.2 Asymptotic limit $\varepsilon \to 0$

Our next goal is to let $\varepsilon \to 0$ in (7.7), and, in particular, in the remainder $Q_{\varepsilon,\delta}$.

1. We have

$$\int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{u}_\varepsilon \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x$$

$$= \int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{U}_{\varepsilon,\delta} \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x + \int_\Omega n_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \cdot \nabla_x \mathbf{v} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x,$$

where

$$\int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{U}_{\varepsilon,\delta} \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x$$

$$= \int_\Omega n_\varepsilon \nabla_x \Pi \cdot (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \ \mathrm{d}x + \int_\Omega n_\varepsilon \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \mathbf{v} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x,$$

and

$$\int_\Omega n_\varepsilon \nabla_x \Pi \cdot (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \ \mathrm{d}x = \int_\Omega n_\varepsilon \nabla_x \Pi \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x - \varepsilon \int_\Omega N_\varepsilon \nabla_x \Pi \cdot (\mathbf{v} + \nabla_x \Psi_{\varepsilon,\delta}) \ \mathrm{d}x + \overline{n} \int_\Omega \nabla_x \Pi \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x.$$

Since

$$n_\varepsilon \mathbf{u}_\varepsilon = [\sqrt{n_\varepsilon}]_{\mathrm{ess}} \sqrt{n_\varepsilon} \mathbf{u}_\varepsilon + [\sqrt{n_\varepsilon}]_{\mathrm{res}} \sqrt{n_\varepsilon} \mathbf{u}_\varepsilon,$$

where, by virtue of estimates (4.1), (4.3),

$$[\sqrt{n_\varepsilon}]_{\mathrm{ess}} \sqrt{n_\varepsilon} \mathbf{u}_\varepsilon \to \overline{n} \mathbf{U} \ \text{weakly-(*) in} \ L^\infty(0,T; L^2(\Omega; R^3)), \ \mathrm{div}_x \mathbf{U} = 0,$$

while

$$[\sqrt{n_\varepsilon}]_{\mathrm{res}} \sqrt{n_\varepsilon} \mathbf{u}_\varepsilon \to 0 \ \text{in} \ L^\infty(0,T; L^{5/4}(\Omega)),$$

we get

$$\mathrm{ess} \sup_{t \in (0, T_{\mathrm{loc}})} \left| \int_\Omega n_\varepsilon \nabla_x \Pi \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x \right| \to 0 \ \text{for} \ \varepsilon \to 0.$$

Similarly, we use (7.15) to observe that

$$\left\{ t \mapsto \int_\Omega \nabla_x \Pi \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \right\} \to 0 \ \text{in} \ L^2(0, T_{\mathrm{loc}}) \ \text{as} \ \varepsilon \to 0$$

for any fixed $\delta > 0$.

Thus we conclude that

$$\left| \int_\Omega n_\varepsilon \Big(\partial_t \mathbf{v} + \mathbf{u}_\varepsilon \nabla_x \mathbf{v} + \frac{1}{\tau} \mathbf{v} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \right| \le c \int_\Omega n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 \ \mathrm{d}x + h^1_{\varepsilon,\delta}, \tag{7.17}$$

where

$$h^1_{\varepsilon,\delta} \to 0 \ \text{in} \ L^2(0, T_{\mathrm{loc}}) \ \text{as} \ \varepsilon \to 0. \tag{7.18}$$

2. Taking advantage of the fact that $\mathrm{div}_x \mathbf{v} = 0$ we can write

$$\int_\Omega n_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x = \varepsilon \int_\Omega N_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x,$$

where $\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta}$ can be expressed by means of equation (7.4). Using (7.15), (7.16) we conclude that

$$\left| \int_\Omega n_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x \right| = h^2_{\varepsilon,\delta}, \tag{7.19}$$

with

$$h^2_{\varepsilon,\delta} \to 0 \text{ in } L^2(0, T_{\mathrm{loc}}) \text{ as } \varepsilon \to 0. \tag{7.20}$$

3. Using (7.14), (7.15), we show that

$$\left| \int_\Omega n_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x^2 \Psi_{\varepsilon,\delta} \cdot (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \ \mathrm{d}x \right| + \left| \int_\Omega n_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \right| \tag{7.21}$$

$$+ \left| \int_\Omega (n_\varepsilon - \overline{n}) \mathbf{U}_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \right| + \left| \int_\Omega (n_\varepsilon - \overline{n}) \partial_t |\nabla_x \Psi_{\varepsilon,\delta}|^2 \ \mathrm{d}x \right| \le h^3_{\varepsilon,\delta},$$

with

$$h^3_{\varepsilon,\delta} \to 0 \text{ in } L^2(0, T_{\mathrm{loc}}) \text{ as } \varepsilon \to 0. \tag{7.22}$$

4. Now,

$$\int_\Omega \mathbb{S}_\varepsilon (\nabla_x \mathbf{U}_{\varepsilon,\delta}) : \nabla_x (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \tag{7.23}$$

$$\le \frac{\mu_\varepsilon}{2} \int_\Omega \left| \nabla_x (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) + \nabla_x^t (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) - \frac{2}{3} \mathrm{div}_x (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}) \mathbb{I} \right|^2 \ \mathrm{d}x$$

$$+ c\mu_\varepsilon \int_\Omega |\nabla_x \mathbf{U}_{\varepsilon,\delta}|^2 \ \mathrm{d}x.$$

5. Next, in accordance with (4.3) and (7.14), (7.15),

$$\left| \int_\Omega \Delta \Psi_{\varepsilon,\delta} \left(\frac{P(n_\varepsilon) - P(\overline{n})}{\varepsilon} \frac{n_\varepsilon - \overline{n}}{\varepsilon} - \frac{E(n_\varepsilon, \overline{n})}{\varepsilon^2} \right) \ \mathrm{d}x \right| \tag{7.24}$$

$$+ \left| \frac{\overline{n}}{\varepsilon^2} \int_\Omega \left(P(n_\varepsilon) - \frac{p'(\overline{n})}{\overline{n}} (n_\varepsilon - \overline{n}) - P(\overline{n}) \right) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \right| \le h^4_{\varepsilon,\delta} \to 0 \text{ in } L^2(0, T_{\mathrm{loc}}) \text{ as } \varepsilon \to 0.$$

6. Finally, using (5.19), (6.2), and (6.3), we infer that

$$\left| \int_\Omega N_\varepsilon \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \right| = h^5_{\varepsilon,\delta} \to 0 \text{ in } L^2(0, T) \text{ as } \varepsilon \to 0. \tag{7.25}$$

Using estimates (7.17 - 7.25) in (7.7) we conclude that

$$\int_\Omega \left(\frac{1}{2} n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v} - \nabla_x \Psi_{\varepsilon,\delta}|^2 \right)(s,\cdot)\ \mathrm{d}x \tag{7.26}$$

$$+ \int_\Omega \left(\frac{1}{\varepsilon^2} E(n_\varepsilon, \overline{n}) - p'(\overline{n}) N_\varepsilon s_{\varepsilon,\delta} + p'(\overline{n}) \frac{\overline{n}}{2} |s_{\varepsilon,\delta}|^2 \right)(s,\cdot)\ \mathrm{d}x$$

$$+ \int_\Omega \left(\frac{1}{2} \left| \nabla_x \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \right|^2 + \overline{n} N_\varepsilon \Delta_N^{-1}[s_{\varepsilon,\delta}] + \frac{\overline{n}^2}{2} \left| (-\Delta_n)^{-1/2} [s_{\varepsilon,\delta}] \right|^2 \right)(s,\cdot)\ \mathrm{d}x$$

$$\leq \int_\Omega \left(\frac{1}{2} n_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{H}[\mathbf{u}_0] - \nabla_x [\Psi_{0,\varepsilon}]_\delta|^2 \right)\ \mathrm{d}x$$

$$+ \int_\Omega \left(\frac{1}{\varepsilon^2} E(n_{0,\varepsilon}, \overline{n}) - p'(\overline{n}) N_{0,\varepsilon} [N_{0,\varepsilon}]_\delta + p'(\overline{n}) \frac{\overline{n}}{2} \left| [N_{0,\varepsilon}]_\delta \right|^2 \right)\ \mathrm{d}x$$

$$+ \int_\Omega \left(\frac{1}{2} \left| \nabla_x \left(\frac{\Phi_{0,\varepsilon}}{\varepsilon} \right) \right|^2 + \overline{n} N_{0,\varepsilon} \Delta_N^{-1} [N_{0,\varepsilon}]_\delta + \frac{\overline{n}^2}{2} \left| (-\Delta_n)^{-1/2} [N_{0,\varepsilon}]_\delta \right|^2 \right)\ \mathrm{d}x$$

$$+ c \int_0^s \int_\Omega \left(\frac{1}{2} n_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v} - \nabla_x \Psi_{\varepsilon,\delta}|^2 \right)\ \mathrm{d}x\ \mathrm{d}t + \int_0^s h^6_{\varepsilon,\delta}\ \mathrm{d}t$$

where

$$h^6_{\varepsilon,\delta} \to 0 \text{ in } L^2(0, T_{\mathrm{loc}}) \text{ as } \varepsilon \to 0. \tag{7.27}$$

Now, we claim that

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \sqrt{\frac{E(n_\varepsilon, \overline{n})}{\varepsilon^2}} - \sqrt{\frac{p'(\overline{n})}{2\overline{n}}} N_\varepsilon \right\|_{L^{4/3}(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Indeed we have

$$\sqrt{\frac{E(n_\varepsilon, \overline{n})}{\varepsilon^2}} - \sqrt{\frac{p'(\overline{n})}{2\overline{n}}} N_\varepsilon$$

$$= \left[\left(\frac{H(n_\varepsilon) - H'(\overline{n})(n_\varepsilon - \overline{n}) - H(\overline{n})}{\varepsilon^2} \right)^{1/2} - \left(\frac{1}{2} H''(\overline{n}) \frac{(n_\varepsilon - \overline{n})^2}{\varepsilon^2} \right)^{1/2} \right]_{\mathrm{ess}}$$

$$+ \left[\left(\frac{H(n_\varepsilon) - H'(\overline{n})(n_\varepsilon - \overline{n}) - H(\overline{n})}{\varepsilon^2} \right)^{1/2} - \left(\frac{1}{2} H''(\overline{n}) \frac{(n_\varepsilon - \overline{n})^2}{\varepsilon^2} \right)^{1/2} \right]_{\mathrm{res}},$$

where

$$\left[\left(\frac{H(n_\varepsilon) - H'(\overline{n})(n_\varepsilon - \overline{n}) - H(\overline{n})}{\varepsilon^2} \right)^{1/2} - \left(\frac{1}{2} H''(\overline{n}) \frac{(n_\varepsilon - \overline{n})^2}{\varepsilon^2} \right)^{1/2} \right]_{\mathrm{res}} \to 0$$

$$\text{in } L^\infty(0,T; L^q(\Omega)),\ 1 \leq q < 5/3,$$

while

$$\left| \left[\left(\frac{H(n_\varepsilon) - H'(\overline{n})(n_\varepsilon - \overline{n}) - H(\overline{n})}{\varepsilon^2} \right)^{1/2} - \left(\frac{1}{2} H''(\overline{n}) \frac{(n_\varepsilon - \overline{n})^2}{\varepsilon^2} \right)^{1/2} \right]_{\mathrm{ess}} \right|$$

$$\leq \sqrt{\varepsilon}\left[\sqrt{H'''(\xi)\frac{|n_\varepsilon-\overline{n}|^3}{\varepsilon^3}}\right]_{\rm ess} \quad \text{for a certain } \xi\in[\overline{n}/2,2\overline{n}].$$

Consequently, relation (7.26) can be written in the form

$$\int_\Omega \left(\frac{1}{2}n_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{v}-\nabla_x\Psi_{\varepsilon,\delta}|^2\right)(s,\cdot)\ {\rm d}x \tag{7.28}$$

$$+\int_\Omega\left(\sqrt{\frac{E(n_\varepsilon,\overline{n})}{\varepsilon^2}}-\sqrt{\frac{p'(\overline{n})\overline{n}}{2}}s_{\varepsilon,\delta}\right)^2(s,\cdot)\ {\rm d}x+\int_\Omega\frac{1}{2}\left|(-\Delta)^{-1/2}\left[s_{\varepsilon,\delta}-N_\varepsilon\right]\right|^2(s,\cdot)\ {\rm d}x$$

$$\leq\int_\Omega\left(\frac{1}{2}n_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}-\mathbf{H}[\mathbf{u}_0]-\nabla_x[\Psi_{0,\varepsilon}]_\delta|^2\right)\ {\rm d}x$$

$$+\int_\Omega\left(\frac{1}{\varepsilon^2}E(n_{0,\varepsilon},\overline{n})-p'(\overline{n})N_{0,\varepsilon}[N_{0,\varepsilon}]_\delta+p'(\overline{n})\frac{\overline{n}}{2}\Big|[N_{0,\varepsilon}]_\delta\Big|^2\right)\ {\rm d}x$$

$$+\int_\Omega\left(\frac{1}{2}\left|\nabla_x\left(\frac{\Phi_{0,\varepsilon}}{\varepsilon}\right)\right|^2+\overline{n}N_{0,\varepsilon}\Delta_N^{-1}[N_{0,\varepsilon}]_\delta+\frac{\overline{n}^2}{2}\left|(-\Delta_n)^{-1/2}[N_{0,\varepsilon}]_\delta\right|^2\right)\ {\rm d}x$$

$$+c\int_0^s\int_\Omega\left(\frac{1}{2}n_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{v}-\nabla_x\Psi_{\varepsilon,\delta}|^2\right)\ {\rm d}x\ {\rm d}t+\int_0^s h^7_{\varepsilon,\delta}\ {\rm d}t$$

with

$$h^7_{\varepsilon,\delta}\to 0 \text{ in } L^2(0,T_{\rm loc}) \text{ as } \varepsilon\to 0.$$

Applying Gronwall's lemma we therefore get

$$\int_\Omega \left(\frac{1}{2}n_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{v}-\nabla_x\Psi_{\varepsilon,\delta}|^2\right)(s,\cdot)\ {\rm d}x$$

$$+\int_\Omega\left(\sqrt{\frac{E(n_\varepsilon,\overline{n})}{\varepsilon^2}}-\sqrt{\frac{p'(\overline{n})\overline{n}}{2}}s_{\varepsilon,\delta}\right)^2(s,\cdot)\ {\rm d}x+\int_\Omega\frac{1}{2}\left|(-\Delta)^{-1/2}\left[s_{\varepsilon,\delta}-N_\varepsilon\right]\right|^2(s,\cdot)\ {\rm d}x$$

$$\leq R_{\varepsilon,\delta}(s)+\left(C\int_0^s R_{\varepsilon,\delta}(t)e^{-Ct}{\rm d}t\right)e^{Cs},$$

where

$$R_{\varepsilon,\delta}(s)=\int_\Omega\left(\frac{1}{2}n_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}-\mathbf{H}[\mathbf{u}_0]-\nabla_x[\Psi_{0,\varepsilon}]_\delta|^2\right)\ {\rm d}x$$

$$+\int_\Omega\left(\frac{1}{\varepsilon^2}E(n_{0,\varepsilon},\overline{n})-p'(\overline{n})N_{0,\varepsilon}[N_{0,\varepsilon}]_\delta+p'(\overline{n})\frac{\overline{n}}{2}\Big|[N_{0,\varepsilon}]_\delta\Big|^2\right)\ {\rm d}x$$

$$+\int_\Omega\left(\frac{1}{2}\left|\nabla_x\left(\frac{\Phi_{0,\varepsilon}}{\varepsilon}\right)\right|^2+\overline{n}N_{0,\varepsilon}\Delta_N^{-1}[N_{0,\varepsilon}]_\delta+\frac{\overline{n}^2}{2}\left|(-\Delta_n)^{-1/2}[N_{0,\varepsilon}]_\delta\right|^2\right)\ {\rm d}x+\int_0^s h^7_{\varepsilon,\delta}\ {\rm d}t.$$

Thus, letting $\varepsilon\to 0$ we obtain

$$\operatorname*{ess\,sup}_{\varepsilon\to 0}\sup_{s\in(0,T_{\rm loc})}\int_\Omega\left(\frac{1}{2}n_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{v}-\nabla_x\Psi_{\varepsilon,\delta}|^2\right)(s,\cdot)\ {\rm d}x\leq\chi(\delta), \tag{7.29}$$

where the function χ is determined in terms of the initial data, and $\chi(\delta)\to 0$ as $\delta\to 0$.

7.3 Asymptotic limit $\delta \to 0$

Letting $\delta \to 0$ in (7.29) we may infer that

$$\mathbf{H}[\sqrt{n}_\varepsilon \, \mathbf{u}_\varepsilon] \to \sqrt{\overline{n}}\, \mathbf{v} \text{ in } L^2(0, T_{\rm loc}; L^2(K; R^3)), \tag{7.30}$$

$$\mathbf{H}^\perp[\sqrt{n}_\varepsilon \, \mathbf{u}_\varepsilon] \to 0 \text{ in } L^2(0, T_{\rm loc}; L^2(K; R^3)) \tag{7.31}$$

for any compact $K \subset \Omega$.

Relations (7.30), (7.31) complete the proof of Theorem 3.2.

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